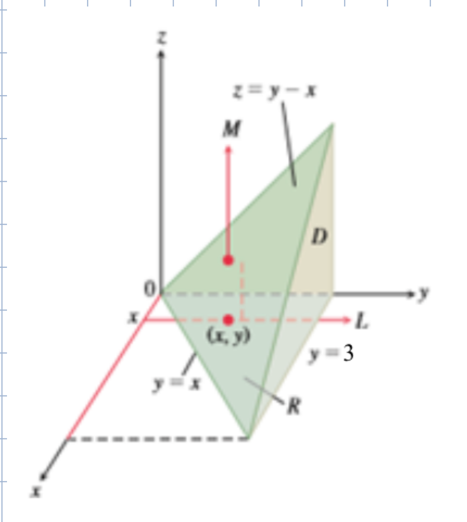
15.5.1 Find the volume of the tetrahedron shown using the order dz dx dy. The tetrahedron is bounded by the planes $z=0$, $y=3$, $x=0$, and $z=y-x$.



the volume of a enclosed, bounded region D in space is : $V = \iiint_D dv$

$0 \le z \le y - x$, $0 \le x \le y$, $0 \le y \le 3$.

thus, the triple integral is :

$$\int_0^3\int_0^y\int_0^{y-x} dz\,dx\,dy = \int_0^3\int_0^y [z]_0^{y-x}\,dx\,dy$$

$$= \int_0^3\int_0^y (y-x)\,dx\,dy$$

$$= \int_0^3 \left[yx - \frac{x^2}{2}\right]_0^y dy$$

$$= \int_0^3 \left(y^2 - \frac{y^2}{2}\right) dy$$

$$= \left[\frac{y^3}{3} - \frac{y^3}{6}\right]_0^3$$

$$= \left(\frac{3^3}{3} - \frac{3^3}{6}\right)$$

$$= \frac{9}{2}$$

therefore, the triple integral has the value $\frac{9}{2}$ unit(s)^3.

15.5.7 Evaluate the integral $\int_0^2\int_0^5\int_0^3 (x^2+y^2+z^2)\,dz\,dy\,dx$

$$\int_0^2\int_0^5\int_0^3 (x^2+y^2+z^2)\,dz\,dy\,dx = \int_0^2\int_0^5 \left[x^2z + y^2z + \frac{z^3}{3}\right]_0^3 dy\,dx$$

$$= \int_0^2\int_0^5 \left[x^2 3 + y^2 3 + \frac{3^3}{3}\right]_0^3 dy\,dx$$

$$= \int_0^2\int_0^5 (3x^2 + 3y^2 + 9)\,dy\,dx$$

$$= \int_0^2 \left[3x^2y + y^3 + 9y\right]_0^5 dx$$

$$= \int_0^2 \left[3x^2 5 + 5^3 + 9 5\right] dx$$

$$= \int_0^2 (15x^2 + 170)\,dx$$

$$= \left[5x^3 + 170x\right]_0^2$$

$$= \left[5\,2^3 + 170\,2\right]_0^2$$

$$= 380$$

15.5.15 Evaluate the iterated integral $\int_{-3}^{7}\int_{0}^{2x}\int_{y}^{x-3} dz\, dy\, dx$

$$\int_{-3}^{7}\int_{0}^{2x}\int_{y}^{x-3} dz\, dy\, dx = \int_{-3}^{7}\int_{0}^{2x} \Big[z\Big]_{y}^{x-3} dy\, dx$$

$$= \int_{-3}^{7}\int_{0}^{2x} (x-3-y)\, dy\, dx$$

$$= \int_{-3}^{7} \left[xy - 3y - \frac{y^2}{2}\right]_{0}^{2x} dx$$

$$= \int_{-3}^{7} \left[x\,2x - 3\,2x - \frac{2x^2}{2}\right] dx$$

$$= \int_{-3}^{7} \left(\cancel{2x^2} - 6x - \cancel{2x^2}\right) dx$$

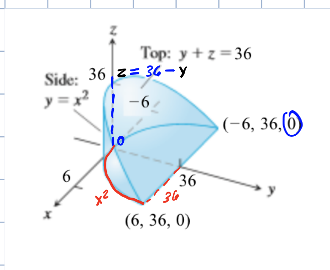
$$= \Big[-3x^2\Big]_{-3}^{7}$$

$$= \left(-3\,7^2\right) - \left(-3\,{-3}^2\right)$$

$$= (-147 + 27)$$

$$= -120$$

15.5.21 Here is the region of integration of the integral $\int_{-6}^{6}\int_{x^2}^{36}\int_{0}^{36-y} dz\, dy\, dx$



Rewrite the integral as an equivalent integral in the following orders.

a. dy dz dx
b. dy dx dz
c. dx dy dz
d. dx dz dy
e. dz dx dy

a. dy dz dx

since $x^2 \le y \le 36$ and $0 \le z \le 36 - y$,
change the limits of y in terms of z and the limits of z in terms of x.

substitute $y = x^2$ and $y = 36$ in $z = 36 - y$

$z = 36 - y$ $\qquad$ $z = 36 - y$

$z = 36 - x^2$ $\qquad$ $z = 36 - 36$

$\qquad\qquad\qquad$ $z = 0$

therefore, $0 \le z \le 36 - x^2$

rewrite $z = 36 - y$ in terms of y.

$y = 36 - z$

therefore, $x^2 \le y \le 36 - z$

$$\int_{-6}^{6}\int_{x^2}^{36}\int_{0}^{36-y} dz\, dy\, dx = \int_{-6}^{6}\int_{0}^{36-x^2}\int_{x^2}^{36-z} dy\, dz\, dx$$

b. dy dx dz

since y doesn't change: $x^2 \le y \le 36 - z$

Find limits of x in terms of z.

Solve $y = x^2$ for x

$x^2 = y$

$\sqrt{x^2} = \sqrt{36 - y}$

$x = \pm\sqrt{36 - z}$

so, $-\sqrt{36-z} \le x \le \sqrt{36-z}$

when $x = 0$ the value of z is: 36

so, $0 \le z \le 36$.

therefore,

$$\int_{-6}^{6}\int_{x^2}^{36}\int_{0}^{36-y} dz\,dy\,dx = \int_{0}^{36}\int_{-\sqrt{36-z}}^{\sqrt{36-z}}\int_{x^2}^{36-z} dy\,dx\,dz$$

C. dx dy dz

since $x^2 \le y \le 36$ x needs to be in terms of y

solve $y = x^2$ for x

$\sqrt{x^2} = \sqrt{y}$

$x = \pm\sqrt{y}$

so, $-\sqrt{y} \le x \le \sqrt{y}$

solve $z = 36 - y$ for y

$z = 36 - y$

$y = 36 - z$

so, $0 \le y \le 36 - z$

when $y = 0$ the value of z is: 36

so, $0 \le z \le 36$.

therefore,

$$\int_{-6}^{6}\int_{x^2}^{36}\int_{0}^{36-y} dz\,dy\,dx = \int_{0}^{36}\int_{0}^{36-z}\int_{-\sqrt{y}}^{\sqrt{y}} dx\,dy\,dz$$

d. dx dz dy

solve $y = x^2$ for x dx doesn't change.

$\sqrt{x^2} = \sqrt{y}$

$x = \pm\sqrt{y}$

so, $-\sqrt{y} \le x \le \sqrt{y}$

therefore,

$$\int_{-6}^{6}\int_{x^2}^{36}\int_{0}^{36-y} dz\,dy\,dx = \int_{0}^{36}\int_{0}^{36-y}\int_{-\sqrt{y}}^{\sqrt{y}} dx\,dz\,dy$$

z already in terms of y: $0 \le z \le 36 - y$

solve $z = 36 - y$ for y when $z = 0$.

$z = 36 - y$

$0 = 36 - y$

$y = 36$

so, $0 \le y \le 36$

15.5.39 Find the average value of $F(x, y, z) = x^2 + y^2 + z^2$ over the rectangular solid in the first octant bounded by the planes $x = 3$, $y = 3$, $z = 1$.

The average value of a function over a region D in space is defined by the following formula.

$$\text{Average value} = \frac{1}{\text{volume of } D}\iiint_D F(x,y,z)\,dV$$

The volume of the rectangular solid is (3)(3)(1) = 9.

$$\begin{aligned}
\text{Average Value} &= \frac{1}{9}\int_0^1\int_0^3\int_0^3 (x^2+y^2+z^2)\,dx\,dy\,dz \\
&= \frac{1}{9}\int_0^1\int_0^3 \left[\frac{x^3}{3} + xy^2 + xz^2\right]_0^3 dy\,dz \\
&= \frac{1}{9}\int_0^1\int_0^3 \left[\frac{3^3}{3} + 3y^2 + 3z^2\right]_0^3 dy\,dz \\
&= \frac{1}{9}\int_0^1\int_0^3 (9 + 3y^2 + 3z^2)\,dy\,dz \\
&= \frac{1}{9}\int_0^1 \left[9y + y^3 + 3yz^2\right]_0^3 dz \\
&= \frac{1}{9}\int_0^1 \left[9 3 + 3^3 + 3 3z^2\right]_0^3 dz \\
&= \frac{1}{9}\int_0^1 (27 + 27 + 9z^2)\,dz \\
&= \frac{1}{9}\int_0^1 (54 + 9z^2)\,dz \\
&= \frac{1}{9}\left[54z + 3z^3\right]_0^1 \\
&= \frac{1}{9}\left[54 1 + 3 1^3\right]_0^1 \\
&= \frac{1}{9}(57) \\
&= \frac{19}{3}
\end{aligned}$$

15.5.35 Find the volume of the region cut from the solid elliptical cylinder $x^2 + 4y^2 \le 4$ by the xy-plane $z = x + 2$.

$$x^2 + 4y^2 \le 4$$
$$x^2 + 4y^2 = 4$$

Find the y-limits

$$x^2 + 4y^2 = 4$$
$$4y^2 = 4 - x^2$$
$$y^2 = \frac{4 - x^2}{4}$$
$$\sqrt{y^2} = \pm\sqrt{\frac{4 - x^2}{4}}$$
$$y = \pm\frac{\sqrt{4 - x^2}}{2}$$

$$-\frac{\sqrt{4 - x^2}}{2} \le y \le \frac{\sqrt{4 - x^2}}{2}$$

Let $y = 0$ in $x^2 + 4y^2 = 4$

$$x^2 + 4(0)^2 = 4$$
$$x^2 = 4$$
$$\sqrt{x^2} = \pm\sqrt{4}$$
$$x = \pm 2$$

$$-2 \le x \le 2$$

z-limits

$$0 \le z \le x + 2$$

therefore,

$$V = \int_{-2}^{2}\int_{-\frac{\sqrt{4-x^2}}{2}}^{\frac{\sqrt{4-x^2}}{2}}\int_{0}^{x+2} dz\,dy\,dx = \int_{-2}^{2}\int_{-\frac{\sqrt{4-x^2}}{2}}^{\frac{\sqrt{4-x^2}}{2}} \left[z\right]_0^{x+2} dy\,dx$$

$$= \int_{-2}^{2}\int_{-\frac{\sqrt{4-x^2}}{2}}^{\frac{\sqrt{4-x^2}}{2}} (x+2)\,dy\,dx$$

$$= \int_{-2}^{2} (x+2)\left[y\right]_{-\frac{\sqrt{4-x^2}}{2}}^{\frac{\sqrt{4-x^2}}{2}} dx$$

$$= \int_{-2}^{2} (x+2)\left[\frac{\sqrt{4-x^2}}{2} + \frac{\sqrt{4-x^2}}{2}\right] dx$$

$$= \int_{-2}^{2} (x+2)\left[\cancel{2}\left(\frac{\sqrt{4-x^2}}{\cancel{2}}\right)\right] dx$$

$$= \int_{-2}^{2} (x+2)\left(\sqrt{4-x^2}\right) dx$$

$$= \int_{-2}^{2} \left[x\left(\sqrt{4-x^2}\right) + 2\left(\sqrt{4-x^2}\right)\right] dx$$

$$\int x\sqrt{a^2-x^2}\,dx = -\frac{1}{3}\left(\sqrt{a^2-x^2}\right)^3 + C$$
$$\int \sqrt{a^2-x^2}\,dx = \frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a} + C$$

$$= \left[-\frac{1}{3}\left(\sqrt{4-x^2}\right)^3\right]_{-2}^{2} + \left[\cancel{2}\left(\frac{x}{\cancel{2}}\sqrt{4-x^2} + \frac{4}{\cancel{2}}\sin^{-1}\frac{x}{2}\right)\right]_{-2}^{2}$$

$$= -\frac{1}{3}\left[(4-x^2)^{\frac{1}{2}\cdot 3}\right]_{-2}^{2} + \left[x\sqrt{4-x^2} + 4\sin^{-1}\frac{x}{2}\right]_{-2}^{2}$$

$$= -\frac{1}{3}\left[(4-x^2)^{\frac{3}{2}}\right]_{-2}^{2} + \left[x\sqrt{4-x^2} + 4\sin^{-1}\frac{x}{2}\right]_{-2}^{2}$$

$$= -\frac{1}{3}\left[(4-2^2)^{\frac{3}{2}} - (4--2^2)^{\frac{3}{2}}\right] + \left[\left(2\sqrt{4-2^2} + 4\sin^{-1}\frac{2}{2}\right) - \left(-2\sqrt{4--2^2} + 4\sin^{-1}\frac{-2}{2}\right)\right]$$

$$= -\frac{1}{3}\left[0\right] + \left[(0 + 4\sin^{-1}1 + 0 - 4\sin^{-1}-1\right]$$

$$= \left[4\sin^{-1}1 - 4\sin^{-1}-1\right]$$

$$= \cancel{4}\left(\frac{\pi}{\cancel{2}}\right) - \cancel{4}\left(-\frac{\pi}{\cancel{2}}\right)$$

$$= 2\pi + 2\pi$$

$$= 4\pi$$

15.8.3

a. Solve the system u = 2x + 3y, v = x + 4y for x and y in terms of u and v. Then find the value of the Jacobian $\frac{\partial(x,y)}{\partial(u,v)}$

Solve the given equations for x and y in terms of u and v by substitution. First solve the equation v = x + 4y for x.

$$v = x + 4y$$
$$x = v - 4y$$

Substitute the expression for x into the first equation and solve for y in terms of u and v.

$$u = 2x + 3y$$
$$u = 2(v - 4y) + 3y$$
$$u = 2v - 8y + 3y$$
$$u = 2v - 5y$$
$$5y = 2v - u$$
$$y = \frac{2v - u}{5}$$

Substitute the expression for y into the equation for x and solve for x in terms of u and v.

$$x = v - 4y$$
$$x = v - 4\left(\frac{2v - u}{5}\right)$$
$$x = v - \frac{8v + 4u}{5} = \frac{4u - 3v}{5}$$

Find the Jacobian, J(u, v), for the transformation.

$$J(u,v) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial y}{\partial u}\frac{\partial x}{\partial v}$$

Find each derivative. Begin by differentiating x with respect to u on the left and v on the right.

$$x = \frac{4u - 3v}{5} \qquad x = \frac{4u - 3v}{5}$$
$$\frac{\partial x}{\partial u} = \frac{4}{5} \qquad \frac{\partial x}{\partial v} = -\frac{3}{5}$$

Differentiate the expression for y with respect to u on the left and v on the right.

$$y = \frac{2v - u}{5} \qquad y = \frac{2v - u}{5}$$
$$\frac{\partial y}{\partial u} = -\frac{1}{5} \qquad \frac{\partial y}{\partial v} = \frac{2}{5}$$

Evaluate the Jacobian.

$$J(u,v) = \left(\frac{4}{5}\right)\left(\frac{2}{5}\right) - \left(-\frac{1}{5}\right)\left(-\frac{3}{5}\right)$$
$$= \frac{1}{5}$$

b. Find the image under the transformation $u = 2x + 3y$, $v = x + 4y$ of the triangular region in the xy-plane bounded by the x-axis, the y-axis, and the line $x + y = 1$. Sketch the transformed image in the uv-plane.

Find the image under the transformation by substituting $x = \frac{4u - 3v}{5}$ and $y = \frac{2v - u}{5}$ into the equations for the boundaries and writing the equations in terms of u and v. Then graph the image in the uv-plane.

Transform the equation for the x-axis, $y = 0$.

$$y = 0$$
$$\frac{2v - u}{5} = 0$$
$$2v - u = 0$$
$$2v = u$$
$$v = \frac{u}{2}$$

Transform the equation for the y-axis, $x = 0$.

$$x = 0$$
$$\frac{4u - 3v}{5} = 0$$
$$-3v = 0$$
$$4u = v$$
$$v = \frac{4}{3}u$$

Transform the equation $x + y = 1$.

$$x + y = 1$$
$$\frac{4u - 3v}{5} + \frac{2v - u}{5} = 1$$
$$4u - 3v + 2v - u = 5$$
$$3u - v = 5$$
$$-1 \div -v = 5 - 3u \div -1$$
$$v = -5 + 3u$$

Graph the lines $v = \frac{u}{2}$, $v = \frac{4}{3}u$ and $v = -5 + 3u$ for the image under the transformation $u = 2x + 3y$, $v = x + 4y$.

15.8.7 Use the transformation $u = 4x + 3y$, $v = x + 3y$ to evaluate the given integral for the region R bounded by the lines $y = -\frac{4}{3}x + 1$, $y = -\frac{4}{3}x + 4$, $y = -\frac{1}{3}x$ and $y = -\frac{1}{3}x + 2$.

$$\iint_R (4x^2 + 15xy + 9y^2)\, dx\, dy$$

$$\iint_R f(x,y)\, dx\, dy = \iint_G f(g(u,v), h(u,v))\, |J(u,v)|\, du\, dv \quad \text{where} \quad J(u,v) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial y}{\partial u}\frac{\partial x}{\partial v}$$

write $$\iint_R (4x^2 + 15xy + 9y^2)\, dx\, dy = \iint_R (4x + 3y)(x + 3y)\, dx\, dy$$

since the transformation is $u = 4x + 3y$ $v = x + 3y$

so, $$\iint_R (4x + 3y)(x + 3y)\, dx\, dy = \iint_R uv\, |J(u,v)|\, du\, dv$$

since $u = 4x + 3y$ $v = x + 3y$ solve for x and y.

$$v = x + 3y$$
$$x = v - 3y$$

$$u = 4x + 3y$$
$$u = 4(v - 3y) + 3y$$
$$u = 4v - 12y + 3y$$
$$u = 4v - 9y$$
$$9y = 4v - u$$
$$y = \frac{4v - u}{9}$$

$$x = v - 3y$$
$$x = v - 3\left(\frac{4v - u}{9}\right)$$
$$x = v - \frac{4v + u}{3}$$
$$x = \frac{u - v}{3}$$

Find the Jacobian, J(u, v), for the transformation. $J(u,v) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial y}{\partial u}\frac{\partial x}{\partial v}$

$$= \begin{vmatrix} \frac{\partial x}{\partial u}\left[\frac{u-v}{3}\right] & \frac{\partial x}{\partial v}\left[\frac{u-v}{3}\right] \\ \frac{\partial y}{\partial u}\left[\frac{4v-u}{9}\right] & \frac{\partial y}{\partial v}\left[\frac{4v-u}{9}\right] \end{vmatrix} = \begin{vmatrix} \frac{1}{3} & -\frac{1}{3} \\ -\frac{1}{9} & \frac{4}{9} \end{vmatrix} = \left(\frac{1}{3}\right)\left(\frac{4}{9}\right) - \left(-\frac{1}{9}\right)\left(-\frac{1}{3}\right) = \frac{1}{9}$$

therefore,

$$\int_R\int (4x+3y)(x+3y)\,dx\,dy = \int_R\int uv\,|J(u,v)|\,du\,dv$$

$$= \frac{1}{9}\int_G\int uv\,du\,dv$$

transform the equation

Transform the equation $y = -\frac{4}{3}x + 1$

$$\frac{4v-u}{9} = -\frac{4}{3}\left(\frac{u-v}{3}\right) + 1$$

$$9\cdot\ \frac{4v-u}{9} = -\frac{4u+4v}{9} + 1\ \cdot 9$$

$$\cancel{4v} - u = -4u + \cancel{4v} + 9$$

$$-9 = -4u + u$$

$$-9 = -3u$$

$$u = 3$$

Transform the equation $y = -\frac{4}{3}x + 4$

$$\frac{4v-u}{9} = -\frac{4}{3}\left(\frac{u-v}{3}\right) + 4$$

$$9\cdot\ \frac{4v-u}{9} = -\frac{4u+4v}{9} + 4\ \cdot 9$$

$$\cancel{4v} - u = -4u + \cancel{4v} + 36$$

$$-36 = -4u + u$$

$$-36 = -3u$$

$$u = 12$$

Transform the equation $y = -\frac{1}{3}x$

$$\frac{4v-u}{9} = -\frac{1}{3}\left(\frac{u-v}{3}\right)$$

$$9\cdot\ \frac{4v-u}{9} = -\frac{u+v}{9}\ \cdot 9$$

$$4v - \cancel{u} = -\cancel{u} + v$$

$$3v = 0$$

$$v = 0$$

Transform the equation $y = -\frac{1}{3}x + 2$

$$\frac{4v-u}{9} = -\frac{1}{3}\left(\frac{u-v}{3}\right) + 2$$

$$9\cdot\ \frac{4v-u}{9} = -\frac{u+v}{9} + 2\ \cdot 9$$

$$4v - \cancel{u} = -\cancel{u} + v + 18$$

$$3v = 18$$

$$v = 6$$

Hence,

$$\frac{1}{9}\int_G\int uv\,du\,dv = \frac{1}{9}\int_0^6\int_3^{12} uv\,du\,dv$$

$$= \frac{1}{9}\int_0^6 \left[\frac{u^2}{2}\right]_3^{12} v\,dv$$

$$= \frac{1}{9}\int_0^6 \left(\frac{135}{2}\right) v\,dv$$

$$= \frac{15}{2}\left[\frac{v^2}{2}\right]_0^6$$

$$= \frac{15}{2}\left[\frac{v^2}{2}\right]_0^6$$

$$= \frac{15}{2}(18) = 135$$

15.8.9 Let R be the region in the first quadrant of the xy-plane bounded by the hyperbolas $xy = 1$, $xy = 25$, and the lines $y = x$, $y = 16x$. Use the transformation $x = \frac{u}{v}$, $y = uv$ with $u > 0$ and $v > 0$ to rewrite the integral over an appropriate region G in the uv-plane. Then evaluate the uv-integral over G.

$$\int\int_R \left(\sqrt{\frac{y}{x}} + \sqrt{xy}\right) dx\, dy \qquad \int\int_R f(x,y)\, dx\, dy = \int\int_G f(g(u,v),\, h(u,v))\ |J(u,v)|\, du\, dv$$

xy-equations for the boundary of R	$x = \frac{u}{v}$, $y = uv$ Corresponding uv-equations for the boundary of G	Simplified uv-equations
$xy = 1$	$\frac{u}{v} \cdot uv = 1$	$\frac{u}{\cancel{v}} \cdot u\cancel{v} = 1$ $u^2 = 1$ $u = 1$
$xy = 25$	$\frac{u}{v} \cdot uv = 25$	$\frac{u}{\cancel{v}} \cdot u\cancel{v} = 25$ $u^2 = 25$ $u = 5$
$y = x$	$uv = \frac{u}{v}$	$uv = \frac{u}{v}$ $uv^2 = u$ $v^2 = 1$ $v = 1$
$y = 16x$	$uv = 16\frac{u}{v}$	$uv = 16\frac{u}{v}$ $uv^2 = 16u$ $v^2 = 16$ $v = 4$

Find the Jacobian, J(u, v), for the transformation. $x = \frac{u}{v}$, $y = uv$.

$$J(u,v) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial y}{\partial u}\frac{\partial x}{\partial v}$$

$$u \cdot \frac{1}{v} = u \cdot v^{-1} = u \cdot -v^{-2} = u \cdot -\frac{1}{v^2} = -\frac{u}{v^2}$$

$$= \begin{vmatrix} \frac{\partial x}{\partial u}\left[\frac{u}{v}\right] & \frac{\partial x}{\partial v}\left[\frac{u}{v}\right] \\ \frac{\partial y}{\partial u}\left[uv\right] & \frac{\partial y}{\partial v}\left[uv\right] \end{vmatrix} = \begin{vmatrix} \frac{1}{v} & -\frac{u}{v^2} \\ v & u \end{vmatrix} = \left(\frac{1}{v}\right)(u) - (v)\left(-\frac{u}{v^2}\right) = \left(\frac{u}{v}\right) - \left(-\frac{u}{v}\right) = 2\frac{u}{v}$$

thus,

$$\frac{uv}{1} \cdot \frac{v}{u} = v^2$$

$$\int\int_R \left(\sqrt{\frac{y}{x}} + \sqrt{xy}\right) dx\, dy = \int_1^4\int_1^5 \left(\sqrt{\frac{uv}{\frac{u}{v}}} + \sqrt{\frac{u}{v} \cdot u}\right)\left(2\frac{u}{v}\right) du\, dv = \int_1^4\int_1^5 \left(\sqrt{v^2} + \sqrt{u^2}\right)\left(2\frac{u}{v}\right) du\, dv$$

$$= \int_1^4\int_1^5 (v+u)\left(2\frac{u}{v}\right) du\, dv = 2\int_1^4\int_1^5 \left(u + \frac{u^2}{v}\right) du\, dv$$

$$= 2\int_1^4 \left[u^2 + \frac{u^3}{3v} \right]_1^5 dv = 2\int_1^4 \left[\left(\frac{5^2}{2} + \frac{5^3}{3v} \right) - \left(\frac{1^2}{2} + \frac{1^3}{3v} \right) \right] dv = 2\int_1^4 \left[\left(\frac{25}{2} + \frac{125}{3v} - \frac{1}{2} - \frac{1}{3v} \right) \right] dv = 2\int_1^4 \left[\left(12 + \frac{124}{3v} \right) \right] dv$$

$$= 2\left(\cdot\frac{124}{3} \left[12v + \ln v \right]_1^4 \right) \text{ since } \int \frac{1}{v} = \ln v = 2\left(\cdot\frac{124}{3} \left[12\cdot 4 + 2\ln 2 - (12\cdot 1 + \ln 1) \right] \right) = 2\left(\cdot\frac{124}{3}\left(48 + 2\ln 2 - 12 - 0 \right) \right) = 2\left(36 + \frac{248}{3}\ln 2 \right) \quad 2\left(\frac{248}{3}\ln 2 + 36 \right)$$

Use the transformation u = x + 2y, v = y − x to evaluate the given integral by first writing it as an integral over a region G in the uv-plane.

$$\int_0^{\frac{2}{3}} \int_y^{2-2y} (x+2y)\, e^{(y-x)}\, dx\, dy$$

Solution:- Given integral ①

$$\int_0^{\frac{2}{3}} \int_y^{2-2y} (x+2y)\, e^{y-x}\, dx\, dy$$

Now, $\int_y^{2-2y} (x+2y)\, e^{y-x}\, dx$

Apply integration by parts

Formula:- $\int uv\,dx = u\int v\,dx - \int \left[\frac{d}{dx}u \int v\,dx \right] dx$

Here $u = x+2y \quad v = e^{y-x}$

$$\Rightarrow \int (x+2y)\, e^{y-x} dx = (x+2y)\int e^{y-x} dx - \int \left[\frac{d}{dx}(x+2y) \int e^{y-x} dx \right] dx$$

$$= (x+2y)\, \frac{e^{y-x}}{-1} - \int (1)\left(-e^{y-x}\right) dx$$

$$= -(x+2y)e^{y-x} + \int e^{y-x}\, dx$$

$$= -(x+2y)e^{y-x} - e^{y-x}$$

Then

$$\int_y^{2-2y} (x+2y)e^{y-x} dx = -\left[(x+2y)e^{y-x} + e^{y-x} \right]_y^{2-2y}$$

$$= -\left[(2-2y+2y)e^{y-2+2y} + e^{y-2+2y} - (3y)e^{0} - e^{0} \right]$$

$$= -\left[2e^{3y-2} + e^{3y-2} - 3y - 1 \right] = 3y - 3e^{3y-2} + 1$$

Now $\int_0^{\frac{2}{3}} \left(3y - 3e^{3y-2} + 1 \right) dy$

②

$$\Rightarrow \int_0^{\frac{2}{3}} 3y\,dy - \int_0^{\frac{2}{3}} 3e^{3y-2} dy + \int_0^{\frac{2}{3}} 1\,dy \rightarrow ①$$

Now, $\int_0^{\frac{2}{3}} 3y\,dy = 3\left(\frac{y^2}{2} \right)_0^{\frac{2}{3}} = \frac{3}{2}\left[\left(\frac{2}{3}\right)^2 - 0 \right] = \frac{3}{2} \times \frac{4}{9} = \frac{2}{3}$

And $\int_0^{\frac{2}{3}} 3e^{3y-2} dy$ let $3y-2 = t$, $3dy = dt$

$$\Rightarrow \int_0^{\frac{2}{3}} e^t\, dt = \left(e^t \right)_0^{\frac{2}{3}}$$

$$= \left(e^{3y-2} \right)_0^{\frac{2}{3}} = e^{2-2} - e^{-2} = 1 - e^{-2}$$

And $\int_0^{\frac{2}{3}} 1\,dy = (y)_0^{\frac{2}{3}} = \frac{2}{3}$

Now Substitute all these in ①

$$\Rightarrow \frac{2}{3} - (1 - e^{-2}) + \frac{2}{3}$$

$$\Rightarrow \frac{4}{3} - 1 + e^{-2}$$

$$\Rightarrow \frac{1}{3} + \frac{1}{e^2}$$

$$\therefore \int_0^{\frac{2}{3}} \int_y^{2-2y} (x+2y)\, e^{y-x}\, dx\, dy = \boxed{\frac{1}{3} + \frac{1}{e^2}}$$

Evaluate $\int_C (x+y)ds$ where C is the straight-line segment x = 3t, y = (6 − 3t), z = 0 from (0,6,0) to (6,0,0).

$\int_C (x+y)ds = 36\sqrt{2}$

Solution:-

$x = 3t$

$y = (6-3t)$

$z = 0$

from (0,6,0) to (6,0,0)

when $x = 3t = 0 \quad t = 0$

$3t = 6 \quad t = 2$

Unusually $0 \le t \le 2$

C is $\vec{r}(t) \rightarrow 3t\hat{i} + (6-3t)\hat{j} + 0\hat{k}$

$\vec{r}'(t) \rightarrow 3\hat{i} - 3\hat{j}$

$\|\vec{r}'(t)\| = \sqrt{9+9} = 3\sqrt{2} = |\vec{v}(t)|$

So

$$\int_C (x+y)ds$$

$$= \int_0^2 (x+y)\, \|\vec{r}'\|\, dt \qquad \vec{v}(t)$$

$ds = |\vec{v}(t)|dt$

$ds = \|\vec{r}'(t)\|dt$

$x = 3t$

$y = (6-3t)$

$$\int_0^2 (\cancel{3t} + 6 - \cancel{3t}) \times 3\sqrt{2}\,dt = \int_0^2 18\sqrt{2}\,dt$$

$$= 18\sqrt{2}\,[t]_0^2$$

$$= 18\sqrt{2}\,(2-0)$$

$$= 36\sqrt{2}$$

Ans

⊗ Given curve is $\vec{r}(t) = 2t\hat{i} + t\hat{j} + (8-2t)\hat{k}$, $0 \le t \le 1$

$\therefore \left(\dot{\vec{r}}(t)\right) = 2\hat{i} + \hat{j} - 2\hat{k}$

$\therefore |\dot{\vec{r}}(t)| = \sqrt{4+1+4} = 3$

$\therefore ds = |\dot{\vec{r}}(t)|\,dt = 3\,dt$

$$\int_C f(x,y,z)\,ds = \int_a^b f(g(t),h(t),k(t))\,|v(t)|\,dt$$

$$\therefore \int_C (xy + x + z)\,ds = \int_{t=0}^{1} \{2t\cdot t + 2t + (8-2t)\}\cdot 3\,dt$$

$$\int_C (x,y,z)\,ds = \quad 2t^2 + 8 \quad 3(2t^2+8)\,dt$$

$$= \int_0^1 3(2t^2 + \cancel{t} + 8)\,dt \qquad 3t$$

$$= 3\left[\frac{2}{3}t^3 + \cancel{\quad} + 8t\right]_0^1$$

$$= \boxed{\cancel{\frac{49}{2}}} \qquad 26$$

Integrate $f(x,y,z) = x + \sqrt{y} - z^4$ over the path from (0,0,0) to (1,1,1) given by

C_1: $\mathbf{r}(t) = t\,\mathbf{i} + t^2\mathbf{j}$, $0 \le t \le 1$

C_2: $\mathbf{r}(t) = \mathbf{i} + \mathbf{j} + t\mathbf{k}$, $0 \le t \le 1$.

use the equation $\displaystyle\int_C (x+\sqrt{y}-z^4)\,ds = \int_{C_1} (x+\sqrt{y}-z^4)\,ds + \int_{C_2} (x+\sqrt{y}-z^4)\,ds$

For C_1: $r(t) = t i + t^2 j$, $0 \le t \le 1$

$x(t) = t$, $y(t) = t^2$, $z(t) = 0$

$x'(t) = 1$, $y'(t) = 2t$, $z'(t) = 0$

$ds = |v'(t)|$

$= \sqrt{1^2 + (2t)^2 + 0^2}\,dt$

$\sqrt{1 + 4t^2}\,dt$

From starting / ending point

From parametrization

$$\int_C (x+\sqrt{y}-z^4)\,ds = \int_0^1 (t + t - 0^4)\sqrt{1+4t^2}\,dt$$

$$= \int_0^1 2t\sqrt{1+4t^2}\,dt$$

$u = 1 + 4t^2$

$du = 8t\,dt$

$t\,dt = \frac{1}{8}du$

$\therefore \int t\frac{\sqrt{u}}{8}du = \int \frac{\sqrt{u}}{8}du = \frac{u^{\frac12+1}}{8\cdot\frac12+1} = \frac{u^{\frac32}}{12}$

$\frac{(1+4t^2)^{3/2}}{12}$

$$= 2\left[\frac{(1+4t^2)^{3/2}}{12}\right]$$

$$= \frac{1}{6}\left[(1+4t^2)^{3/2}\right]_0^1 = \frac{1}{6}(5)^{\frac32} - 1 = \frac{1}{6}(5\sqrt{5}-1) \quad ①$$

For C_2: $r(t) = i + j + tk$, $0 \le t \le 1$

$x(t) = 1$, $y(t) = 1$, $z(t) = t$

$x'(t) = 0$, $y'(t) = 0$, $z'(t) = 1$

$ds = |v'(t)|$

$= \sqrt{1^2}\,dt$

$= 1\,dt$

$$\int_C (x+\sqrt{y}-z^4)\,ds = \int_0^1 (1+1-t^4)\,dt = \int_0^1 (2-t^4)\,dt$$

$$= (2-t^4)\,dt = \left[2t - \frac{t^5}{5}\right]_0^1 = \frac{9}{5} \quad ②$$

thus, $\displaystyle\int_C (x+\sqrt{y}-z^4)\,ds = \frac{1}{6}(5\sqrt{5}-1) + \frac{9}{5} = \frac{5\sqrt{5}}{6} - \frac{1}{6} + \frac{9}{5} = \frac{5\sqrt{5}}{6} + \frac{49}{30}$

ANS.

16.1.27 Integrate f over the given curve.

$f(x,y) = \frac{x^3}{y}, \quad C: \; y = \frac{x^2}{2}, \quad 0 \le x \le 1$

the equation of the smooth parameterization of C is :

$r(x) = xi + yj$

$r(x) = xi + \frac{x^2}{2} j$

$r'(x) = i + xj$

$|r'(x)| = \sqrt{1+x^2} \; |v(t)|$

$f(x,y) = \frac{x^3}{y}$

$= \frac{x^3}{\frac{x^2}{2}} = 2x$

$\int_C f(x,y,z)\,ds = \int_a^b f(g(t),h(t),k(t))\,|v(t)|\,dt$

$\int_C f(x,y)\,ds = \int_0^1 2x\sqrt{1+x^2}\,dx$

u $1+x^2$

du $2x\,dx$

$x=0 \quad u=1$

$x=1 \quad u=2$

$= \int_1^2 \sqrt{u}\,du$

$= \frac{u^{\frac{3}{2}}}{\frac{3}{2}}$

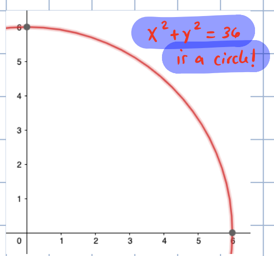
$= \left[\frac{2}{3}u^{\frac{3}{2}}\right]_1^2$

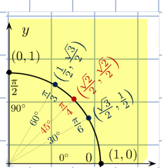
$= \left[\frac{2}{3}2^{\frac{3}{2}} - \left(\frac{2}{3}1^{\frac{3}{2}}\right)\right]$

$= \frac{2 \cdot 2\sqrt{2}}{3} - \frac{2 \cdot 1\sqrt{1}}{3}$

$= \frac{4\sqrt{2}-2}{3}$

16.1.29 Integrate $f(x,y) = x+y$ over the curve $C: x^2+y^2 = 36$ in the first quadrant from $(6,0)$ to $(0,6)$.





$x^2 + y^2 = 36$

$r(t) = (6\cos t)i + (6\sin t)j \qquad 0 \le t \le \frac{\pi}{2}$

$\int_C f(x,y)\,ds = \int_a^b f(g(t),h(t))\,|v(t)|\,dt$

differentiated.

$v(t) = (-6\sin t)i + (6\cos t)j$

$|v(t)| = \sqrt{(-6\sin t)^2 + (6\cos t)^2}$

$= \sqrt{36\sin^2 t + 36\cos^2 t}$

$= \sqrt{36(\sin^2 t + \cos^2 t)}$

$= 6\,dt$

$\int_C f(x,y)\,ds = \int_0^{\frac{\pi}{2}} f(6\cos t, 6\sin t)\cdot 6\,dt$

$= \int_0^{\frac{\pi}{2}} (6\cos t + 6\sin t)\cdot 6\,dt$

$= \int_0^{\frac{\pi}{2}} 6(\cos t + \sin t)\cdot 6\,dt$

$= 36\int_0^{\frac{\pi}{2}} (\cos t + \sin t)\,dt$

$= 36\left[\sin t - \cos t\right]_0^{\frac{\pi}{2}}$

$= 36\left[\sin\frac{\pi}{2} - \cos\frac{\pi}{2} - (\sin 0 - \cos 0)\right]$

$= 36\left[1 - 0 - (0-1)\right] = 36 \cdot 2 = 72$

16.1.29 Find the mass of a wire that lies along the curve $r(t) = (t^2 - 5)j + 2t\,k$, $0 \le t \le 3$, if the density is $\delta = \frac{3}{2}t$.

Step 1 of 3

Consider the curve $r(t) = \left(t^2 \boxed{-5}\right)j + 2tk, 0 \le t \le \boxed{3}$ and the density is $\delta = \frac{3}{2}t$.

The objective is to find the mass of the curve.

Comment

Step 2 of 3

The equation of the curve is, $r(t) = \left(t^2 \boxed{-5}\right)j + 2tk, 0 \le t \le \boxed{3}$

Differentiate with respect to *t*.

$\frac{dr}{dt} = 2tj + 2k$ Use $\frac{d}{dx}x^n = nx^{n-1}$

$\left|\frac{dr}{dt}\right| = \sqrt{4t^2 + 4}$ Since $|x| = \sqrt{x}$

$= 2\sqrt{t^2 + 1}$

Comment

Step 3 of 3

Assume $M = \int_c \delta(x,y,z)\,ds = \int_c \delta(x,y,z)\left|\frac{dr}{dt}\right| \cdot dt$ $ds = \left|\frac{dr}{dt}\right| \cdot dt$

$M = \int_C \delta(x,y,z)\,ds = \int_C \delta(x,y,z)\,|v(t)|\,dt$

$M = \int_0^3 \delta(t)\left[2\sqrt{t^2+1}\right]dt$

$= \int_0^3 \left(\frac{3}{2}t\right)\left[2\sqrt{t^2+1}\right]dt$

$= \int_0^3 \left(\frac{3}{\cancel{2}}t\right)\left[\cancel{2}(t^2+1)^{\frac{1}{2}}\right]dt$ ......(1)

Consider $(t^2+1)^{\frac{1}{2}} = v$. $v = (t^2+1)^{\frac{1}{2}}$ $t = 3$ $v = \sqrt{10}$

$dv = \frac{1}{2}(t^2+1)^{-\frac{1}{2}} \cdot 2t$ $t = 0$ $v = 1$

$(t^2+1)^{\frac{1}{2}} = v$

$\frac{1}{2}(t^2+1)^{\frac{1}{2}-1}(2t) = dv$ $dv = \frac{2t}{2(t^2+1)^{1/2}}$ $dv = \frac{t}{\sqrt{t^2+1}}$

$\frac{t}{\sqrt{t^2+1}} = dv$

$t = \sqrt{t^2+1}\,dv$ $t = \sqrt{t^2+1}\,dv$

$t = v\,dv$ $t = v\,dv$

This implies,

$M = 3\int_1^{\sqrt{10}} v \cdot v\,dv$

$= 3\int_0^1 v^2\,dv$ Since $\int x^n dx = \frac{x^{n+1}}{n+1}$

$= \left[\left(\sqrt{2}\right)^2 - 1\right]$ $2^{\frac{1}{2}\cdot 2} = 2\sqrt{2} - 1$

$= \boxed{2\sqrt{2} - 1}$ $10\sqrt{10} - 1$

$\frac{v^3}{3} = \frac{(\sqrt{10})^3}{3} = \frac{10^{\frac{1}{2}\cdot 3}}{3} = \frac{10^{\frac{3}{2}}}{3} = \frac{10\sqrt{10}}{3}$

$= \frac{1^3}{3} = \frac{1}{3}$

$= \frac{10\sqrt{10}}{3} - \frac{1}{3}$

$= \frac{10\sqrt{10} - 1}{\cancel{3}} \cdot \cancel{3}$